Lévy Processes

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INTRODUCTION

Lévy processes are random processes on Euclidean space that are stochastically continuous and have stationary independent increments. They, and their stochastic integrals, have become useful tools in a variety of nonparametric statistical and environmetric applications including density estimation, survival analysis, regression, and spatial modeling.

At times we may be willing to assume that some observed random lifetimes T_i come from a particular parametric family of distributions, like the Weibull or lognormal or gamma families, so that inference about their distribution reduces to the problem of estimating parameters. In some other problems, especially those in which we fear that the actual distribution has features like multi-modality, heavy tails, etc. that make the commonly-used distributions inappropriate, we may prefer to use a nonparametric modeling approach in which the survival function

$$S(t) \equiv \mathsf{P}[T_i > t], \qquad 0 \le t < \infty$$

is treated as an entirely unknown function. In a similar fashion we may at times be reluctant to postulate that the relationship between quantities of interest Y_i and observable predictor variables ξ_i is linear or even of any specific form, and may instead prefer a nonparametric regression equation of the form

$$Y_i = G(\xi_i) + \epsilon_i$$

with uncorrelated mean-zero errors ϵ_i but with an entirely unknown regression function $G(x)$.

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A Bayesian approach to either of these problems would begin with the construction of a probability distribution on possible survival or regression functions. A common and fruitful way to proceed with such a construction is to begin with an increasing random function X_t with stationary independent increments, and set $S(t) = \exp(-X_t)$ (so that X_t may be interpreted as cumulative hazard), or $G(t) = X_t$ (here non-monotone functions X_t may also be of interest). The Bayesian approach would proceed by calculating or simulating the *posterior* or conditional distribution of X_t (and hence of $S(t)$ or $G(x)$), given the data.

There is increasing interest in extending and applying spatial statistical methods to environmetrical problems; examples include the problem of estimating, interpolating, or predicting the concentration $C(t)$ of an environmental contaminant over a two-dimensional region $t \in \mathcal{R} \subset \mathbb{R}^2_+$. It is rare for a parametric model (linear or otherwise) to be satisfactory in such problems, but it is possible to express uncertainty about contaminant concentrations by modelling $C(t)$ as a nonnegative *random field*, a random process indexed by points in the plane.

SII PROCESSES

A real-valued *Lévy Process* is a random process X_t taking values in $\mathbb{R}$ that is *stochastically-continuous*, meaning that $X_s \to X_t$ in probability as $s \to t$ for each fixed t, and that has *stationary independent increments*, meaning that for any $n \in \mathbb{N}$ and any numbers $-\infty < t_0 < t_1 < \cdots < t_n < \infty$, the increments $(X_{t_{j+1}} - X_{t_j})$ are mutually independent real-valued random variables whose probability distributions depend only on the differences $(t_{j+1} - t_j)$ and not on t_j's themselves. It is customary to require that the sample paths be right-continuous; by stochastic continuity, this entails no real loss of generality. There are natural generalizations to vector-valued processes and to random fields indexed by points in the plane or higher-dimensional spaces, but here we illustrate the ideas with the one-dimensional case.

For any $s < t$ and $n \in \mathbb{N}$ we can write the increment $(X_t - X_s)$ as the sum of n independent identically-distributed random variables, so by the celebrated theorem of Lévy [7] and (independently) Khinchine [6], the characteristic function $\phi_{s,t}(\omega) \equiv \mathsf{E}\big\{ \exp\big[i\omega(X_t - X_s)\big]\big\}$ may be written in the form

$$\exp\left\{ i(t-s)\omega a - (t-s)\omega^2\sigma^2/2 + (t-s)\int_{\mathbb{R}} \left[e^{i\omega u} - 1 - i\omega h(u)\right] \nu(du) \right\} \quad (1)$$

for some numbers $a \in \mathbb{R}$ and $\sigma > 0$, and a positive measure $\nu(du)$ (called the "Lévy measure") on $\mathbb{R}$ satisfying the integrability condition

$$\int_{\mathbb{R}} (1 \wedge u^2)\nu(du) < \infty, \tag{2}$$

where the notation $1 \wedge u^2$ denotes the minimum of 1 and u^2 and where $h(u)$ denotes an arbitrary bounded continuous function (like $x/(1+x^2)$ or $\sin(x)$) satisfying $h(u) = u + \mathcal{O}(u^2)$ near $u = 0$. Evidently $\phi_{s,t}(\omega) = \phi_{0,1}(\omega)^{t-s}$; henceforth we consider only $s = 0$ and $t = 1$, and omit them from the notation.

EXAMPLES

The zero-mean Gaussian Wiener process $w_t \sim \mathsf{No}(0, t)$ is an example with continuous paths, of course, as is Brownian motion with drift $X_t = at + \sigma w_t$ (with drift $a \in \mathbb{R}$ and scale $\sigma^2 > 0$); any other example may be written as the sum of two independent processes, Brownian motion with drift and a pure-jump process with $a = \sigma = 0$. The Lévy measure $\nu(E)$ has a simple interpretation as the rate at which the process takes jumps of size $u \in E$, for any set $E \subset \mathbb{R}$. The integrability condition (2) guarantees that $\nu([-\epsilon, \epsilon]^c) < \infty$ for any number $\epsilon > 0$, so the (Poisson-distributed) number of jumps in finite time $t < \infty$ whose absolute value exceed ϵ is always finite.

The Cauchy process with rate α (starting at zero) has probability density function $f(x) = \alpha/\pi(\alpha^2 + x^2)$ at time $t = 1$, with characteristic function $\phi_C(\omega) = e^{-\alpha|\omega|}$ and Lévy measure $\nu_C(du) = \alpha\, u^{-2}\, du$; more generally the symmetric stable distribution of index $0 < \xi < 2$ and rate $\alpha > 0$, with log characteristic function proportional to $-\alpha|\omega|^{\xi}$, has Lévy measure $\nu_S(du) = \alpha\xi|u|^{-1-\xi}\, du$. These are examples of Lévy processes with infinitely many jumps in every time interval, since $\nu(\mathbb{R}) = \infty$; still the process is always finite, as a consequence of the integrability condition (2).

Lévy processes are increasing if (and only if) $a \geq 0$, $\sigma = 0$, and $\nu(du)$ is concentrated on $(0, \infty)$ and satisfies the stricter integrability condition

$$\int_{\mathbb{R}} (1 \wedge |u|)\nu(du) < \infty. \tag{3}$$

Examples include the Poisson with rate λ, with Lévy measure $\nu_P(du) = \lambda\delta_1(du)$ (a point mass of size λ at $u = 1$), the gamma process, with Lévy measure $\nu_\Gamma(du) = \alpha e^{-\lambda u} u^{-1} du$, and the one-sided (or "fully skewed") stable process with $\nu_{S_+}(du) = \alpha\xi u^{-1-\xi} du$ for $0 < \xi < 1$ (the integrability condition (2) would fail for $\xi \geq 1$). Other increasing examples include the beta process, with Lévy measure $\nu_\beta(du) = \beta(1-u)^{\beta-1}\, u^{-1} du$ on the unit interval $(0,1)$ and the simple homogeneous process, with Lévy measure $\nu_H(du) = \alpha e^{-\beta u}(1 - e^{-u})^{-1}\, du$. All of these are of emerging importance in Bayesian hierarchical nonparametric and semiparametric modeling [4, 10]. The one-sided stable processes have been used as time series innovations [8], the beta for survival analysis [5], and the Poisson and gamma process for point process intensities [11], for example.

PROPERTIES

Their independent increments ensure that all Lévy processes have the Markov property, i.e., that their future and past developments at every time t are conditionally independent, given the present value. A Lévy process will only be a *martingale* (a process whose increments $(X_t - X_s)$ for $s < t$ have conditional expectation zero, given $\{X_u : u \leq s\}$) if $a = 0$ and if $\nu(du)$ is symmetric and satisfies integrability condition (3) to ensure that $\mathsf{E}|X_t| < \infty$, but every Lévy process is a *semimartingale* (the sum of a martingale and a processes with finite-variation paths [9, $p.\,313$]), so stochastic integration theory for Lévy processes is routine.

Although Lévy processes may have infinitely many jumps in finite time, the integrability condition (2) ensures that the quadratic variation $\langle X\rangle_t \equiv \lim_{n\to\infty} \sum_{j<n} (X_{(j+1)t/n} - X_{jt/n})^2$ is finite almost-surely. The absolute variation $\lim_{n\to\infty} \sum_{j<n} |X_{(j+1)t/n} - X_{jt/n}|$ will be finite only if $\sigma = 0$ and if $|u|$ is locally $\nu(du)$-integrable near zero, a condition that fails for the Cauchy example above but is satisfied by the Poisson, gamma, beta, and simple homogeneous examples.

SIMULATION & CONSTRUCTION

For Lévy processes which do have locally finite variation, i.e. which satisfy $\sigma = 0$ and integrability condition (3), (1) can be simplified to $\phi_{s,t}(\omega) = \exp\left\{(t-s)\int_{\mathbb{R}} \left[e^{i\omega u} - 1\right] \nu(du)\right\}$ and the ideas of Lévy [7] lead to a representation of the Lévy process starting at $X_0 = 0$ as the sum

$$X_t = \iint_{\mathbb{R}\times(0,t]} u\, H(du\, ds) \tag{4}$$

of the heights u_m of jumps (u_m, s_m) with $s_m \leq t$ of a planar Poisson measure $H(du\, ds)$ (independent of the Wiener process w_t) with mean $\mathsf{E}[H(du\, ds)] =$

$\nu(du)\, ds$. An explicit implementation for numerical simulation (the Inverse Lévy Measure, or ILM algorithm) is given by Wolpert and Ickstadt [12], generalizing that of Bondesson [2]. The same idea extends to generating Lévy random fields in the plane or higher-dimensional spaces, simply by replacing $s \in \mathbb{R}$ with $s \in \mathbb{R}^p$ and defining the Poisson measure $H(du\, ds)$ on $\mathbb{R}^{1+p}$ (see [11, 12] for details).

A Spatial Example

If pollutant point-sources are located at an unknown Poisson-distributed number M of points s_m distributed uniformly and independently in a two-dimensional region $\mathcal{S} \subset \mathbb{R}^2$, each with some unobserved magnitude $u_m > 0$ that leads to a concentration at points of interest $x \in \mathcal{X} \subset \mathbb{R}^2$ of $u_m\, k(x, s_m)$ for some dispersion kernel $k(x, s)$ on $\mathcal{X} \times \mathcal{S}$, then the aggregate concentration at $x \in \mathcal{X}$ will be

$$\Lambda(x) = \sum u_m\, k(x, s_m),$$

the integral of $k(x, s)$ with respect to the planar Lévy random field with Lévy measure $\nu(du) = \lambda\mu(du)$, where λ is the expected number $\mathsf{E}M$ of point sources and where $\mu(du)$ is the probability distribution for the source intensities u_m. Thus Lévy random fields give a very natural way to model multiple point sources, and to pursue inference (especially in a nonparametric Bayesian approach [3], where the Lévy process serves as a prior distribution for the pollution levels). The simulation methods described in [12], and the Bayesian regression methods described in [1], lead to inference about all unknown features of the model. A number of related examples appear in [10, ch. III].

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